# TCC Week 8

$-\Delta - \frac{\mu}{\delta^2}$ in $\Omega$ – Bounded smooth, $\mu < \frac{1}{4}$

$\beta_- < \beta_+$ – roots of $\boxed{-\beta(\beta-1) = \mu}$

**Theorem (Phragmen-Lindelof alternative)**

1) If $h$ – local small sub-harm. ($\exists\, u_* > 0$ – local super harm $\frac{h}{u_*} \xrightarrow[x \to \partial\Omega]{} 0$)

$$\Rightarrow \limsup_{x \to \partial\Omega} \frac{h}{\delta^{\beta_+}} < +\infty$$

2) If $H$ – local large sub-harm. (not small sub-harm)

$$\Rightarrow \limsup_{x \to \partial\Omega} \frac{H}{\delta^{\beta_-}} > 0$$

Remark:

1) $\mu = 0 \implies \beta_- = 0, \; \beta_+ = 1$

$$\limsup_{x \to \partial\Omega} \frac{h}{\delta} < \infty, \quad \limsup_{x \to \partial\Omega} H > 0.$$

Compare Hopf lemma

2) $\mu < 0 \implies \beta_- < 0, \; \beta_+ > 1$

3) $\mu > 0 \implies \beta_- > 0, \; \beta_+ > 0$ (uniqueness issues!)

# Semilinear problem

$$(*) \quad -\Delta u + \delta^{-s}|u|^{p-1}u = 0 \quad \text{in } \underbrace{\Omega}_{\text{Bounded smooth}}, \quad s \in \mathbb{R}, \quad p > 1$$

Solution of $(*)$ is a $u \in H^1_{loc}(\Omega) \cap C_{loc}(\Omega)$

$$\int_\Omega \nabla u \nabla \varphi + \int_\Omega \underbrace{\delta^{-s} u^p}_{\in L^1_{loc}(\Omega)} \varphi = 0 \quad \forall \varphi \in H^1_c(\Omega)$$

Lemma. $(*)$ has no nonzero solutions in $H^1_0(\Omega)$

◀ Assume $u \in H^1_0(\Omega) \Rightarrow \int_\Omega |\nabla u|^2 + \int_\Omega \delta^{-s}|u|^{p+1} = 0 \Rightarrow$

$\Rightarrow u \equiv 0$ ▶ "No small solutions

Remark: $u$ — small local sub-harmonic to $-\Delta$

$\Rightarrow$ $u \in H_0^1(\Omega)$

Note: $-\Delta u + \delta^{-s} u^p = 0$, $u > 0 \Rightarrow -\Delta u \leq 0$

# Comparison principle for $-\Delta u + \delta^{-s} u^{p-1} = 0$ in $\Omega$

Assume
$$\left.\begin{array}{l} -\Delta v + \delta^{-s} v^p \leq 0 \text{ in } \Omega \\ -\Delta u + \delta^{-s} u^p \geq 0 \text{ in } \Omega \\ \text{"} v \leq u \text{" on } \partial\Omega \end{array}\right\} \Rightarrow v \leq u \text{ in } \Omega$$

in the sence $v, u \in H^1(\Omega)$ and $v \leq u$ on $\partial\Omega$

or $\limsup_{x \to \partial\Omega} (v-u) < 0$.

$$-\Delta(v-u) + \delta^{-s} \underbrace{\frac{v^p - u^p}{v-u}}_{W(x)} (v-u) \leq 0 \text{ in } \Omega$$

Assume $(v-u)^+ \neq 0$. $\Longrightarrow$ $(v-u)^+ \in H_0^1(\Omega)$

$$\int |\nabla (v-u)^+|^2 + \int \delta^{-s} \underbrace{W(x)}_{\geq 0 \text{ in } \operatorname{supp}(v-u)^+} |(v-u)^+|^2 \leq 0 \Rightarrow (v-u)^+ \equiv 0$$

## Keller-Osserman Bound:

If $-\Delta V + \delta^{-s} V^p \le 0$ in $\Omega$

$$\Rightarrow \quad V \le c_* \delta^{-\frac{2-s}{p-1}} \quad \text{in } \Omega!$$

$c_* > 0$ does not depend on $V$!

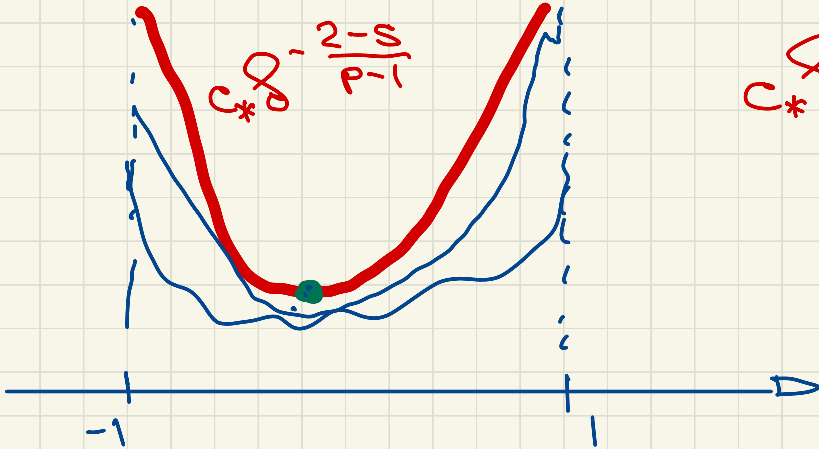
Note that $c\delta^{-\frac{2-s}{p-1}}$ is a local supersol

$$-\Delta\left(c\delta^{-\frac{2-s}{p-1}}\right) + c^p \delta^{-\frac{2-s}{p-1}p-s} \ge 0$$

for $c \ge \gamma_*$ in $\Omega_{\bar{\rho}}$

To make a global supersol. in $\Omega$, use Whitney distance $d_\Omega \in C^2$!

+ comparison principle



$c_* \delta^{-\frac{2-s}{p-1}}$ – universal Barrier!

Corollary. $s>2 \Rightarrow$ no large sub-sol $\Rightarrow$ no large solutions

◀ $s>2 \Rightarrow \delta^{-\frac{2-s}{p-1}} \xrightarrow[x\to\partial\Omega]{} 0$ – contradiction

to $\limsup\limits_{x\to\partial\Omega} v = 0$ ▶

"Moderate solutions" = large solutions for $s<2$.

Thm. Let $s<2$. Then $\forall \eta \in C(\partial\Omega)$, $\eta \geq 0$

$\exists\, u_\eta : \begin{cases} -\Delta u_\eta + u_\eta^p = 0 & \text{in } \Omega \\ u_\eta = \eta & \text{on } \partial\Omega \end{cases}$

$u_\eta$ is known as a moderate solution,

$\eta \not\equiv 0 \Rightarrow u_\eta$ is a large subharmonic

- Take harmonic extension $\tilde{\eta} \in C^2(\Omega)$:

$$\begin{cases} -\Delta \tilde{\eta} = 0 & \text{in } \Omega \\ \tilde{\eta} = \eta & \text{on } \partial\Omega \end{cases}$$

$\tilde{\eta} \in H^1(\Omega)$

Set $u = w + \tilde{\eta}$

$$J(w) = \underbrace{\frac{1}{2}\int_\Omega |\nabla w|^2 + \frac{1}{p+1}\int_\Omega \delta^{-s}(w+\tilde{\eta})^{p+1}}_{\text{convex, coercive: } J(w) \to +\infty \ \ \|w\|_{H^1_0} \to +\infty} : H^1_0(\Omega) \to \mathbb{R}$$

convex, coercive: $J(w) \to +\infty$ $\|w\|_{H^1_0} \to +\infty$

$\Rightarrow$ weakly lower semicont.

$\Rightarrow \exists$ minimiser $w_\eta \in H^1_0(\Omega)$,

$$-\Delta w_\eta + (w_\eta + \tilde{\eta})^p = 0$$

$$-\Delta \underbrace{(w_\eta + \hat{\eta})}_{u_\eta} + \underbrace{(w_\eta + \tilde{\eta})}_{u_\eta}^{p} = 0$$

$$-\Delta u_\eta + u_\eta^{\,p} = 0$$

Boundary blow-up solution $(s<2)$

$\forall M>0$ consider the solution $u_M>0$,

$$\begin{cases} -\Delta u + u^p = 0 & \text{in } \Omega \\ u = M & \text{on } \partial\Omega \end{cases}$$

Then 1) $u_M \leq u_{M'}$ $\forall M<M'$,

2) $u_M \leq c_* \delta^{-\frac{2-s}{p-1}}$ in $\Omega$.

$$\Rightarrow \quad u_\infty := \lim_{M\to\infty} u_M = \sup_{M>0} u_M$$

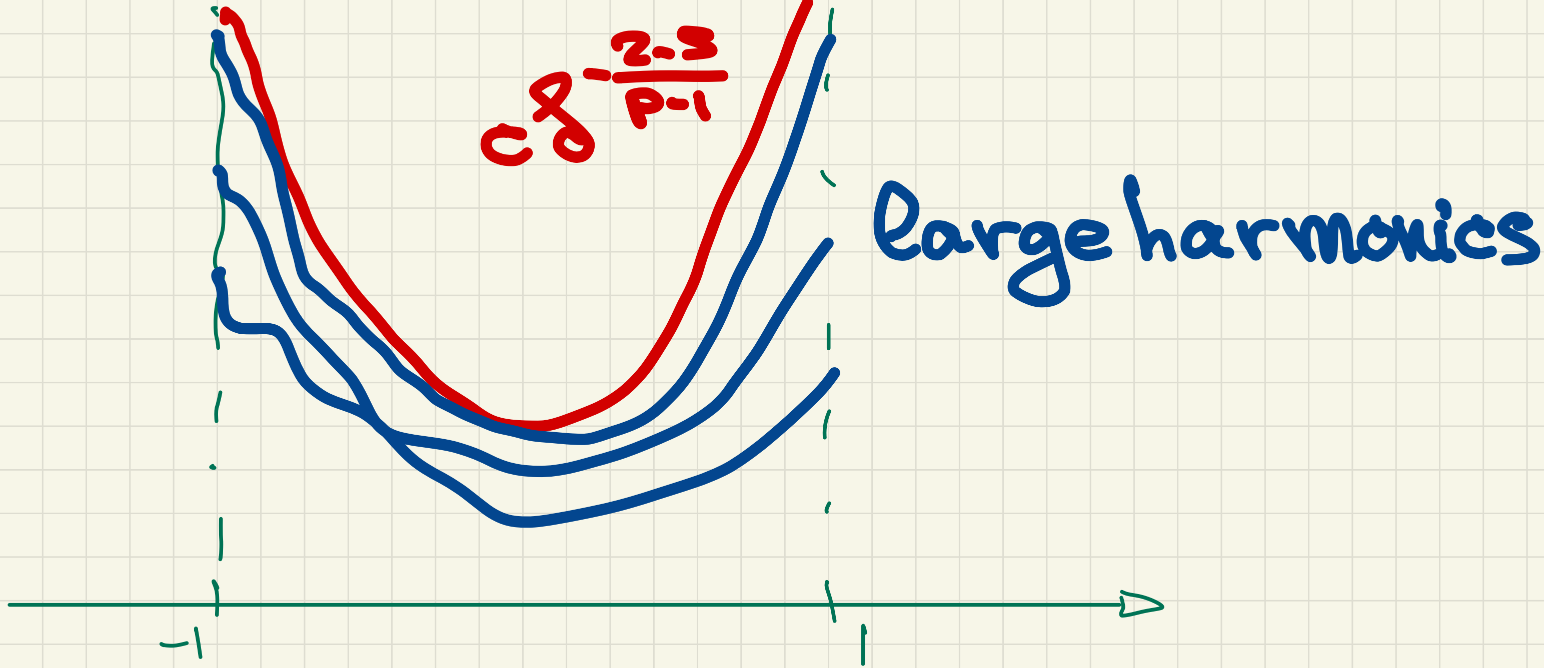
$$u_\infty \leq c_* \delta^{-\frac{2-s}{p-1}}$$

Thm. $s<2 \Rightarrow \exists$ solution $u_\infty > 0$,
"boundary blow-up solution" $u_\infty = +\infty$ on $\partial\Omega$

$$-\Delta u_\infty + u_\infty^p = 0 \quad \text{in } \Omega$$

$$(*) \quad u_\infty = c_\infty \delta^{-\frac{2-s}{p-1}} \left(1 + o(\delta)\right), \quad x \to \partial\Omega$$

Moreover, $u_\infty$ is unique solution that satisfies $(*)$


$c\,\delta^{-\frac{2-3}{p-1}}$
large harmonics
-1
1